ON MEAN FIELD GAMES

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I. INTRODUCTION

• New class of models for the average (Mean Field) behavior of "small" agents (Games) started in the early 2000's by J-M. Lasry and P-L. Lions.

• Requires new mathematical theories.

• Numerous applications : economics, finance, social networks, crowd motions. . .

• Independent introduction of a particular class of MFG models by M. Huang, P.E. Caines and R.P. Malhamé in 2006.

• A research community in expansion : mathematics, economics, finance.

• Some written references but most of the existing mathematical material to be found in the Collège de France videotapes (4 $\times$ 18h) that can be downloaded. . . !

• Combination of Mean Field theories (classical in Physics and Mechanics) and the notion of Nash equilibria in Games theory.

• Nash equilibria for continua of "small" players : a single heterogeneous group of players (adaptations to several groups. . .).

• Interpretation in particular cases (but already general enough!) like process control of McKean-Vlasov. . .

• Each generic player is "rational" i.e. tries to optimize (control) a criterion that depends on the others (the whole group) and the optimal decision affects the behavior of the group (however, this interpretation is limited to some particular situations. . .).

• Huge class of models : agents → particles, no dep. on the group are two extreme particular cases.

II. A REALLY SIMPLE EXAMPLE

• Simple example, not new but gives an idea of the general class of models (other "simple" exs later on).

• E metric space, N players ($1 \leq i \leq N$) choose a position $x_i \in E$ according to a criterion $F_i(X)$ where $X = (x_1, \ldots, x_N) \in E^N$.

• Nash equilibrium : $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_N)$ if for all $1 \leq i \leq N$ $\bar{x}_i$ min over E of $F_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots \bar{x}_N)$.

• Usual difficulties with the notion

• $N \to \infty$? simpler ?

• Indistinguishable players :

$$F_i(X) = F(x_i, \{x_j\}_{j \neq i}), F \text{ sym . in } (x_j)_{j \neq i}$$

- Part of the mathematical theories is about $N \to \infty$:

$$F_i = F(x, m) \quad x \in E \ , \quad m \in \mathcal{P}(E)$$

$$\text{where } \ x = x_i \ , \ m = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$$

- "Thm" : Nash equilibria converge, as $N \to \infty$, to solutions of

$$\text{(MFG)} \qquad \forall x \in \text{ Supp } m, F(x, m) = \inf_{y \in E} F(y, m)$$

- Facts : i) general existence and stability results

ii) uniqueness if $(m \to F(\bullet, m))$ monotone

iii) If $F = \Phi'(m)$, then $(\min_{\mathcal{P}(E)} \Phi)$ yields one solution of MFG.

Example : $E = \mathbb{R}^d, F_i(X) = f(x_i) + g\left(\frac{\#\{j/|x_i - x_j| < \varepsilon\}}{(N-1)|B_\varepsilon|}\right)$

$g \uparrow$ aversion crowds, $g \downarrow$ like crowds

$$F(x,m) = f(x) + g(m * 1_{B_\varepsilon}(x)(|B_\varepsilon|^{-1})$$

$$\varepsilon \to 0 \qquad F(x,m) = f(x) + g(m(x))$$

(MFG) $\qquad \text{supp } m \subset \text{Arg min}\left(f(x) + g(m(x))\right)$

– $g \uparrow$ uniqueness, $g \downarrow$ non uniqueness

$$\min\left\{\int fm + \int G(m) / m \in \mathcal{P}(E)\right\},\ G = \int_0^Z f(s)ds$$

– explicit solution if $g \uparrow$: $m = g^{-1}(\lambda - f), \lambda \in \mathbb{R}\ s.t. \int m = 1$

III. GENERAL STRUCTURE

- Particular case : dynamical problem, horizon T, continuous time and space, Brownian noises (both indep. and common), no intertemporal preference rate, control on drifts (Hamiltonian H), criterion dep. only on m

- $U(x, m, t)$ $(x \in \mathbb{R}^d, m \in \mathcal{P}(\mathbb{R}^d)$ or $\mathcal{M}_+(\mathbb{R}^d), t \in [0, T]$ and $H(x, p, m)$ (convex in $p \in \mathbb{R}^d$)

- MFG master equation

$$\begin{cases} \frac{\partial U}{\partial t} - (\nu + \alpha)\Delta_x U + H(x, \nabla_x U, m) + \\ + \langle \frac{\partial U}{\partial m}, -(\nu + \alpha)\Delta m + \operatorname{div}\,(\frac{\partial H}{\partial p} m) \rangle + \\ -\alpha \frac{\partial U}{\partial m^2}(\nabla m, \nabla m) + 2\alpha \langle \frac{\partial}{\partial m} \nabla_x U, \nabla m \rangle = 0 \end{cases}$$

and $U\,|_{t=0} = U_0(x, m)$ (final cost)

- ν amount of ind. rand. , α amount of common rand.

• ∞ d problem !

• If $\nu = 0$ (ind) : Nash N special case

using $x = x_i, m = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$

• Aggregation/decentralization : IF $H(x, p, m) = H(x, p) + F'(m)$ and $U_0 = \Phi_0'(m)$, then $U = \frac{\partial \Phi}{\partial m}$ solves MFG if Φ solves HJB on $\mathcal{P}(E)$ for the optimal control of a $SPDE$

• Particular case : many extensions and variants ...

IV. THREE PARTICULAR CASES

- ∞ *d* problem in general but reductions to finite *d* in two cases

1. Indep. noises ($\alpha = 0$)

int. along caract. in m yields

$$(\mathrm{MFGi}) \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u, m) = 0 \\ u \mid_{t=0} = U_0(x, m(0)), m \mid_{t=T} = \bar{m} \\ \frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div} \left(\frac{\partial H}{\partial p} m\right) = 0 \end{cases}$$

where $\bar{m}$ is given

FORWARD — BACKWARD system !

contains as particular cases : *HJB*, heat, porous media, *FP*, *V*, *B*, Hartree, semilinear elliptic, barotropic Euler . . .

2. Finite state space ($i \leq i \leq k$)

(MFGf) $\frac{\partial U}{\partial t} + (F(x, U) \,.\, \nabla)\, U = G(x, U), U \,|_{t=0} = U_0$

(no common noise here to simplify . . .)

$$x \in \mathbb{R}^k,\ U \to \mathbb{R}^k,\ F \ \text{ and } \ G : \mathbb{R}^{2k} \to \mathbb{R}^k$$

non-conservative hyperbolic system

Example : If $F = F(U) = H'(U), G \equiv 0$

and if $U_0 = \nabla\varphi_0$ $(\varphi_0 \to \mathbb{R})$ then

– solve HJ

$$\frac{\partial\varphi}{\partial t} + H(\nabla\varphi) = 0\ ,\ \varphi\,|_{t=0} = \varphi_0$$

– take $U = \nabla\varphi$, "U solves" (MFGf) in this case

3. Another point of view

(Ω, F, P) a "rich enough" proba . space $\mathcal{H}$ Hilbert space of L^2 Random Variables

$$\Phi(m) = \Phi(X) \text{ if } \mathcal{L}(X) = m(X \to \mathbb{R}^d)$$

Then MFG may be written as

$$\frac{\partial U}{\partial t} + (\mathcal{F}(X,U).D)U = \mathcal{G}(X,U) + \alpha \Delta_d U$$

$$(+\nu\ D^2 U(G,G)\ \ G \perp F_X)$$

$$\Delta_d U = \Delta_Z\ U(. + Z)|_Z = 0(Z \in \mathbb{R}^d)\ ,$$

where $U : \mathcal{H} \to \mathcal{H}$

Remarks : 1) MFG $U(X) \in F_X, \mathcal{L}(U(X)) = \mathcal{L}(U(Y))$
if $\mathcal{L}(X) = \mathcal{L}(Y)$
2) $U(X) = \nabla_x U(x, \mathcal{L}(X))|_{x=X}$

Allows to prove that the problem is well-posed in the "small".

V. OVERVIEW AND PERSPECTIVES

Lots of questions, partial results exist, many open problems

- Existence/regularity :
  - (MFGi) "simple" if H "smooth" in m (or if H almost linear ...), OK if monotone (Zoom 1)
  - (MFGf) OK if (G, F) mon. on $\mathbb{R}^{2k}$ or small time (Zoom 2)
- Uniqueness : OK if "monotone" or T small ...
- Non existence, non uniqueness, non regularity (!)
- Qualitative properties, stationary states and stability, comparison, cycles ...
- $N \to \infty$ (see above)
- Numerical methods (currently, 3 "general" methods and some particular cases)
- Variants : other noises, several populations ...
- random heterogeneity, partial info ...
- applications (MFG Labs ...)

- intertemporal preference rates ($+\lambda \to \infty$ effective models
- macroscopic limits
- ? Beyond MFG ? (fluctuations, LD, transitions)
- Two more S. examples :
  - at which time will the meeting start ?
  - the (mexican) wave

ZOOM 1

$$(\mathrm{MFGi}) \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = f(x, m) \\ u \mid_{t=0} = U_0(x), m \mid_{t=T} = \bar{m} \\ \frac{\partial m}{\partial t} + \nu \Delta m + \mathrm{div}\, (\frac{\partial H}{\partial p} m) = 0 \end{cases}$$

- $m \mapsto f(\bullet, m)$ smoothing operator
  $\exists$ regular solution
- uniqueness if operator monotone or if T small
- $f(m(x)) \uparrow$: $\exists\, !$ regular solution $\nu > 0$
- $f(m(x)) \uparrow$: if $\nu = 0$ $\;m = f^{-1}(\frac{\partial u}{\partial t} + H(x, \nabla u))$

  equation in m becomes quasilinear elliptic equation of second order ($x \in Q, t \in [0, T]$) with "elliptic" boundary conditions

$$u \mid_{t=0} = U_0(x), \; \frac{\partial u}{\partial t} + H(\nabla u) = f(\bar{m}) \;\text{ if }\; t = T$$

ZOOM 2

$$(\text{MFGf}) \begin{cases} \frac{\partial u}{\partial t} + (F(x,U)\ .\ \nabla)\ U = G(x,U) \quad x \in \mathbb{R}^d \\ U \to \mathbb{R}^d\ ,\ \ U\,|_{t=0} = U_0(x) \end{cases}$$

- shocks (discontinuities of U) in finite time in general
- well-posed problem on $[0, T_{\max})$ ($T_{\max} \leq +\infty$)
- $\exists$!regular solution monotone in x if U_0 monotone and (G, F) monotone of $\mathbb{R}^{2,k}$ in $\mathbb{R}^{2k}(+\dots)$
- $+$ change of unknown functions :

ex. : $\frac{\partial U}{\partial t} + (F(U).\nabla)U = 0$

then $V = F(U)$ solves

$$\frac{\partial V}{\partial t} + (V.\nabla)V = 0$$

max class of regularity

$$\forall \delta > 0, \inf_{x \in \mathbb{R}^d} \operatorname{dist}(Sp(DV_0(x)), (-\infty, \delta]) > 0$$

($V_0 = F(U_0)$ gives the maximum class of regularity $\approx$ composed of 2 monotone applications)

Remark : gives new results of regularity for Hamilton-Jacobi equations of the first order.

VI. MEANINGFUL DATA

- MFG Labs

- Practical expertise and models mainly for "big" data involving "people"

- New models that include classical clustering models in M.L. (K-mean, EM . . .), then algorithms

- No need for euclidean structures or for "a priori" distances

- Why “PEOPLE”

Ex. 1 : Taxis

Ex. 2 : Movies and Fb

People that are “close” will say they like movies that are “close”

$\longrightarrow$ consistency distance - like on items/people

• Even for "pure data" models make sense : data points become agents . . . (in fact lots of terminology from Game Theory in M.L./Data Analysis)

• Clustering : classical K-Mean

set of points $\{x_1, \ldots, x_v\}$ in $\mathbb{R}^d (N >> 1)$

Find K points $y_1, \ldots, y_k$ s.t. $\exists$ partition $(A_1, \ldots, A_K)$ of $\{1, \ldots, N\}$ for which

i) $|y_i - x_j| \leq |y_{i'} - x_j|$, $\forall_j \in A_i$, $\forall i' \neq i$

ii) $y_i = (\# A_i)^{-1} \sum_{j \in A_i} \times j$

MFG INTERPRETATION :

INTRODUCE • A GLOBAL CRITERION

$F(u_1, \dots, u_k)$

Ex : $\min(u_1, \dots, u_k)$

- K value functions $(u_1, \dots, u_k)$
- K "densities" $(m_1, \dots, m_k)$

f being the initial density of "data" (no need to restrict to "discrete" data)

SOLVE MFG : EXAMPLE

$$\rho u_i - \nu \Delta u_i + \frac{1}{2}(\nabla u_i)^2 = F_i(x; m_i)$$

$$\rho m_i - \nu \Delta m_i - div(\nabla u_i m_i) = \rho \, \frac{\partial F}{\partial u_i} \, f$$

ex. $1_{(u_i < \min\limits_{j \neq i} u_j)}$

BACK TO K-Mean

$$F_i = \frac{1+\rho}{2}|x - \frac{\int x m_i}{\int m_i}|^2 - \nu d$$

$$\text{then indeed} \ : \ u_i = \frac{1}{2}|x - y_i|^2, y_i = \frac{\int x m_i}{\int m_i} \ \text{and}$$

$$\int m_i = \int f 1_{(u_i < \min_{j \neq i} u_j)}, \int x m_i = \int x f_i 1_{(u_i < \min_{j \neq i} u_j)}$$

Next, this allows to

- create lots of new models

• smoothe clustering if needed, clusters within clusters, overlaps. . .

• no need for distances, no need for euclidean structure (choose criterion F, class criteria $F_i \to u_i \ldots$)

• transposition to graphs easy (ODE's, massively //)

$$\text{Remark} \ : \quad -\Delta u + |\nabla u|^2 = e^u(+\Delta)e^{-u}$$

$$e^{u_i} \sum_j (e^{-u_j} - e^{-u_i}) = \sum_j (e^{u_i - u_j} - 1)$$

• social networks equilibria : "distance on items" $\longleftrightarrow$ "distance on users" $\longleftrightarrow$ preferences $\longleftrightarrow$ "distances on items". . .